Analysis of Misconceptions in High School Mathematics

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A key aspect of teaching is being able to make appropriate and reasonable accommodations in order to promote access and attainment for all students. An essential skill for teachers and student-teachers when making these appropriate accommodations is being able to identify where students' misconceptions are preventing them from acquiring new conceptual learning. Formative assessment is one tool that allows teachers to identify misconceptions and student weaknesses.

Formative assessment is commonly referred to as assessment "for" learning, in contrast to assessment "of" learning, which reflects summative assessment (Tierney & Charland, 2007). This means that when formative assessment is used purposefully, there is an adjustment of teaching and learning that results from the assessment. A study by Wiliam, Lee, Harrison, and Black (2004) examined the effect of incorporating formative assessment into the classrooms of twenty-four math and science teachers in six different secondary schools. The research results indicated that substantial learning gains resulted when teachers introduced formative assessment into classroom practice. Xiaobao and Yeping's (2008) study of student misconceptions argued that understanding the origins of systematic errors is a vital part of correcting them. Their study concluded that difficulties in learning math are often a result of a student's failure to understand the concepts which form the basis for the procedures they are using.

This becomes a serious inhibitor to learning as mathematics builds on itself. A study by Movshovitz-Hadar, Zaslavsky, and Inbar (1987) undertook a qualitative analysis of errors in high school mathematics by attempting to classify groups of errors made by 11th grade students on a matriculation exam. The resulting "error" classifications were: misusing data, incorrectly translating verbal expressions into mathematical expressions, making logically invalid inferences, applying an improper version of a definition or theorem, having the right solution to the wrong question, or making a mistake in basic skills. These results revealed that a majority of the errors that high school students made in mathematics were not accidental, but instead were derived by a quasi-logical method that made sense to the student. Identification of common errors through examination of completed student work was found to be important because once a student's errors were isolated; a teacher could direct corrective instruction or a remedial plan aimed at that particular error pattern (Riccomini, 2005).

Research on teachers' ability to identify and address student "errors" asserts that because many students who are not proficient in basic math skills demonstrate numerous mathematics misconceptions, it is essential for teachers to recognize various misconceptions when adjusting instruction (Riccomini, 2005). The research of Stefanich and Rokusek (1992) affirms that when a pattern of error was diagnosed and instruction was directed to remediate the incorrect procedure, then new learning could take place quickly, and retention appeared to be long-term. Correspondingly, a study by Wilcox and Zielinski (1997) concluded that assessment helped teachers gain better

insights into their students' understanding, including misconceptions, and therefore helped to better diagnose error patterns and remediate them.

Wiliam (2007), in his investigation of the integration of formative assessment with instruction, focused on relating the way a child solves one problem to how they had solved or might solve other problems, and then using this information to enable teachers to adjust the instruction to meet student needs. Connected with the idea of formative assessment, adjustment of instruction was the critical part of the investigation. The result of this study was that the teachers knew more about the individual students' problem-solving processes, allowing them to modify instruction, which resulted in students doing better in number fact knowledge, understanding, problem solving, and confidence. The current action research study attempts to replicate William's study by investigating the use of formative assessment to categorize and analyze student misconceptions, and then use this data to adapt instruction to accommodate the difficulties of students. Specifically, this study asks: How does analyzing and addressing student misconceptions through formative assessments impact student achievement?

Methodology

Thirty-eight students from two of the student teacher-researcher's high school, non-honors, Algebra II classes participated in the study. The study included nineteen females, nineteen males, seven Hispanics, twenty-five African Americans, and six Caucasians. The researchers analyzed the work of all participating students and five of these participants took part in informal interviews where they were asked to solve math problems orally.

For each lesson in the unit on rational functions, expressions, and equations, the participants were first taught the new concept

or material. The students then completed a short formative assessment quiz on the material. The teacher-researcher graded these daily quizzes, identifying and analyzing specific errors that students made. During the following class period, the errors identified through this process were re-taught based on strategies supported by pedagogical research. At the end of the unit, students completed a unit test that included items that assessed each objective of the larger unit.

Data sources included student work on the following: short one or two question daily (or every other day) quizzes that tested procedural and conceptual understanding and a unit test that measured procedural and conceptual understanding of the completed unit on rational functions, expressions, and equations. A representative group of five volunteer students participated in audio-recorded informal interviews with the researcher to confirm particular misconceptions as they solved problems orally, explaining their thought processes as they worked towards the solution.

Results

The errors were categorized through analysis of the formative assessments. The 265 identified errors consisted of 143 distinct errors, which were classified into error types similar to the classification of errors done by Movshovitz-Hadar, Zaslavsky, and Inbar (1987). This categorization resulted in five predominant error type categories. The error types and the criteria used to organize the analyzed errors into these categories are listed in Table 1.

After categorizing the misconceptions, studies of the five most common error types were compiled, including in each case study the summative test achievement outcomes of participants in whose formative quizzes the misconceptions were found. The following are excerpts from these studies that illustrate each of the error types.

Table 1: Systematic Error Patterns

Error Type	Criteria	Percentage of Total Errors (including repeated errors)
1. Incomplete Answer	Answered portions of the question, but did not provide all the solutions or conclusions required by the question	24.9%
2. Misused Data	Made a conclusion from the data included in the item in a way that was inappropriate, but evidence of correct procedural steps is evident	17.4%
3. Technical Error	Computational error, an error in manipulating elementary algebraic symbols, a careless error, or an error in using processes and skills usually mastered in a prerequisite course	16.2%
4. Error Originating from Misconceptions of Previously Learned Material	Made a mistake in following a procedure or using a skill that is usually mastered at an earlier point in the same course	14.0%
5. Distorted Definition	Altered a definition that is relevant to the solution of the problem	10.6%

Error Type 1: Incomplete Answer

Reciprocal Functions.

Example: State the shifts of the following function:
$$f(x) = \frac{-3}{x+4} + 2$$

Error: Student stated that the function shifted left four and up two, but did not state that the function had both a reflection and a stretch.

Addressing the Error: In order to re-teach, the teacher-researcher reviewed the times there is a reflection of a reciprocal function. This was done by using a graphing calculator to demonstrate this reflection by changing the numerator of the fraction from positive to negative. The graphical representation gave visual emphasis on the effect of having a negative in the numerator. The teacher-researcher reviewed when there is a stretch, and used the graphing calculator to give a visual representation of a stretch transformation.

Reflection on Process: Brown and Burton (1977) suggested it is sometimes necessary to go back and analyze if there was a flaw in the method of teaching that led to an error, and

this is the case for this error. Upon reflection it appeared that the reflection and stretch aspect of shifts had not received enough instructional focus during the original lesson. Therefore, adjustment of instruction was necessary in order to re-teach the concept with emphasis on these two missing elements. The adjustment resulted in an improvement of student achievement.

Error Type 2: Misused Data

Adding and Subtracting Rational Expressions.

Example: Add the following rational expression

$$\frac{1}{4a} + \frac{3}{8a^2}$$

Error: Student found the common denominator correctly, simplifying to get $\frac{2a+3}{8a^2}$, but then tried to cancel out 2a, leaving $\frac{3}{4a}$. The student tried to use the 2a that is a factor of 8a² to cancel out a single term of a sum, which is mathematically incorrect.

Addressing the Error: The teacher-researcher emphasized to the students that they should put sums in parenthesis, and that it is incorrect to cancel out elements in parentheses unless they cancel out everything in the parentheses. Additionally, students spent a day practicing adding and subtracting rational expressions through a jeopardy game.

Reflection on Process: The remedial strategy for this error was based on Brown and Burton's (1977) suggestion that sometimes a clear description of what is going wrong may be a sufficient method of re-teaching. Combining this with a description of how to adjust the problem-solving technique in the remedial plan seemed to result in improvement in student achievement from the quiz to the unit test.

Error Type 3: Technical Errors

Solving Rational Equations.

Example: Solve the equation:

$$\frac{12s+19}{s^2+7s+12} - \frac{3}{s+2} = \frac{5}{s+4}$$

Error: Student did not distribute the negative all the way through the binomial when multiplying -3(s+4) to change the denominator of the second term to the least common denominator (s+2)(s+4). Student instead obtained the product -3s+12.

Addressing the Error: The teacher-researcher reviewed the distributive property. This review included an emphasis on the importance of realizing that when there is subtraction of rational expressions and in which multiplication is required on a term other than the first term to obtain the least common denominator, then the negative should remain with the numerator of that term. The researcher suggested that students rewrite the subtraction as plus a negative in order to remember that the negative is part of the whole numerator.

Reflection on Process: As Cauley and McMillan (2009) suggested, the remedial plan pointed out to students the specific misconception and then demonstrated how the students could adjust their problem-solving technique for the concept. In this case, the concept was not the overall concept of solving rational equations, but rather the prerequisite skill of the distributive property, supporting Brown and Burton's (1977) claim that students make mistakes in procedures because of misunderstandings of previously taught material. In this case the material was previously taught in a different course. However, the remedial strategy did not result in improvement in student achievement in this example.

Error Type 4: Errors Originating from Misconceptions of Previously Learned Material

Multiplying and Dividing Rational Expressions.

Example: Simplify the expression

$$\frac{x^2 + 5x + 4}{x^2 - 16} \cdot \frac{x^2 - 2x - 8}{x^2 + x - 2}$$

Error: Student factored the numerator of the first rational expression incorrectly, allowing the student to be able then to cancel the incorrect binomials.

Addressing the Error: The teacher-researcher reviewed the factoring flow chart that outlines the process that students should follow when factoring. This flow chart included the "easy try," or T-chart, method that is required to factor the numerator of the first rational expression in this problem. The class did several practice problems (guided and individual) that included factoring by the "easy try" method.

Reflection on Process: The remedial strategy incorporates the re-teaching method of reviewing the process step-by-step. Furthermore, the flow chart is a visual representation of the process that students should go through in order to factor correctly. The practice problems, as well as pointing out the source of the student error, follows with Stefanich and Rokusek's (1992) statement that one or two examples pointing out where students were making mistakes will correct the error. The description of the source of the factoring error also aligns with Brown and Burton's (1977) assertion that a clear description of the problem will also correct the error. Yet the combination of these re-teaching strategies in one remedial plan did not result in an improvement in student achievement from the quiz to the unit test.

Error Type 5: Distorted Definition

Rational Functions.

Example: State the domain of the following

rational function: $f(x) = \dfrac{2x}{x - 5}$

Error: Student gave the domain as x is not equal to the root of the rational function,

instead of not equal to the vertical asymptote of the rational function, $x \neq 5$.

Addressing the Error: The teacher-researcher discussed what domain means in terms of domain being all the x-values for which the function is defined, or where if you put in an x you get out a y. She discussed that a vertical asymptote is an invisible line that the function never crosses, therefore if you plugged in that x you would not get out a y, this means that the vertical asymptote is not part of the domain. The teacher-researcher illustrated this concept using the calculator by putting in the rational function and showing that at x=Vertical Asymptote there is an error in the y column. Therefore, for rational functions, x is not equal to the vertical asymptote.

Reflection on Process: The remedial strategy used to re-teach the definition in which the error occurred, the definition of domain, was supported by the suggestions for re-teaching provided by Stefanich and Rokusek (1992), to train students to become aware of reasonable answers. This emphasizes checking work and making sure the answer given is supported; this is an important skill for all students to learn when dealing with any type of mathematics error. After this remedial strategy, which focused solely on the distorted definition of domain that the error revealed, was implemented the percentage of correct responses for items testing this concept increased. There was an improvement in this student's achievement.

After the error type studies were compiled, the mean change in scores from formative daily quizzes to unit tests for the included students in each error type study was calculated. See Table 2.

In three of the five "error type" studies, over fifty percent of the included students improved from the daily quiz items to the unit test items. Additionally, in four of the five studies there was a positive mean change in score from daily quiz items to unit test items. The error type that did not have a positive mean change in score from the formative quiz to the summative test was Technical Errors.

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Table 2: Error Type Studies

Error Type	N students identified and remediated	Mean Change in Score (Quiz→ Test)	% of Students who Improved
1. Incomplete Answer	2	19.775	100%
2. Misused Data	5	15.98	80%
3. Technical Error	6	-11.758	33.3%
4. Error Originating from Misconceptions of Previously Learned Material	3	4.967	33.3%
5. Distorted Definition	2	21.875	100%

Conclusion

The objective of this action research study was to explore the use of formative assessment (daily quizzes), including identification of student weaknesses or lack of understanding, along with

instructional correctives that are different from previous instruction, to help students attain the intended learning goals. Part of the exploration was to determine if the use of this formative assessment focusing on "error types" improved student achievement. The analysis of the data from this study as compared to other "error type" studies supports the conclusion that student achievement does improve when systematic errors are identified from the formative assessment and analyzed. This data was then used to respond to individual learning needs.

The majority of the errors studied resulted in improved student achievement after these errors specifically were addressed with focused instruction guided by the formative

assessment. The error type that did not support improvement was technical errors, which resulted from student inattention, basic math errors, or weakness in prerequisite skills. These were not remediated with concept-focused instruction. Additionally, because technical errors were student specific they were nearly impossible to address to a whole class. The feedback side of formative assessment appeared to be more effective with these types of errors. The other "error type" study that did not find a majority of the students having improved student achievement were errors resulting from misconceptions on previously learned material. These were hard to address with focused instruction in the confines on the time allotted for the current unit.

Therefore, the data suggest that when distinct errors made by individual students were specifically addressed with formative assessment data guiding the instruction to remediate that misconception, then individual student's achievement improved.

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Thus the use of formative assessment data to inform re-teaching is supported, and should be further utilized as a means for continuous individual growth within the scope of pedagogical approaches to the secondary mathematics classroom.

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